MATHEMATICAL ASPECTS OF MODELS FOR HAMILTON HARBOUR, 1978

Ministry of the Environment

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A Developmental Study of Some Numerical and Mathematical Aspects of Models for the Hamilton Harbour

by

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1. Introduction

The effects of different technical solutions to water quality problems in lakes are often studied using a mathematical model of the lake. This model is then transformed into an equivalent numerical model which is usually solved on a large computer.

There are two main problems associated with such numerical models. One is due to the fact that it is not possible to describe exactly all the physical processes which take place in such complex systems. For example the turbulent processes are characterized by empirical parameters; the values of these must then be estimated from measured field data. In the first part of this report we discuss different methods for obtaining estimates of these parameters; and present the conclusions that can be drawn from the application of them to the Hamilton Harbour Model, see the Hamilton Harbour Study (1974). More detailed treatments are given in Appendices A and B.

The second problem is the amount of computing time and hence cost for solving these numerical models. In order to determine the feasibility of reducing cost we are examining some models for which at least part of the solution procedure can be done analytically, so that the expensive numerical treatment would be reduced. In Appendix C such a model is derived.

2. Model validation

The output of a mathematical model is generally in the form of time series of water velocities and waste concentrations at some point of the lake. Such models contain unknown physical parameters, whose values must be adjusted so that the model output agrees reasonably well with the corresponding measured data. This process is called model validation or system parameter identification.

It consists of two parts: a measure of how close the calculated data is to the measured data, and, secondly, a systematic procedure

for adjusting the parameters so that the two time series become closer to each other. The reason that a mathematical definition of closeness of the two sets of data is required, is that in practical situations they will never be identical. In this report we mainly use the square of the differences and maximum difference of the two sets of data as our measures of how close the calculated data is to the measured data.

The main conclusion that can be drawn from the results presented in Appendix B is that with the given time step, 20 sec., and the given space mesh the major features of the flow in Hamilton Harbour are reasonably well modelled, while the more detailed features are not as well reproduced. However it seems likely that with a smaller time step and a finer mesh a better description of these detailed features would be produced at an increased computing cost.

3. Semi-analytic methods for lake modelling

One of the problems with the standard numerical models of lakes, such as those devised by Leenderste (1970), is the large amount of computing time required to solve them. It seems reasonable to expect that one should be able to reduce this cost by using models which can be partially solved analytically so that a computer would only be required for a smaller part of the solution procedure.

The Ekman-type model which is based on the assumption of small Rossby numbers and small horizontal turbulent mixing satisfies this criterion. It has been applied by for example Liggett and Hadjitheodorou (1969) and Gedney and Lick (1972).

A careful study has been done of the derivation of this model, and in Appendix C it is shown that the standard formulation of the boundary conditions must be modified. This should lead to more accurate predictions by the model of the water velocity. We plan to apply a model of this type to the Hamilton Harbour.

Acknowledgements

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Liggett, J.A. and C. Hadjitheodorou, 1969. Circulation in shallow homogeneous lakes. J. Hyd. Div. ASCE, 95, 609.

The work described in this report forms a part of an ongoing study of the Hamilton Harbour. Some of the obtained results are presented in the following publications.

Chan, Y.K., Snodgrass, W.J. and P.T.S. Wong, 1977. Sampler for collecting evolved gasses from sediments. Water Res. 11, 807-809.

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Appendix A

Review of some methods for validation of lake models.

1. Introduction

In the formulation of mathematical models of the circulation and transport of pollutants in lakes it is not possible to exactly describe all the physical processes which take place in such complex systems. For example at present the turbulent processes can only be described approximately by unknown parameters multiplying the gradients of quantities such as mean velocity components and mean pollutant concentrations. Thus one is faced with the problem of attaining values of these empirical parameters. Usually rough estimates can be obtained from laboratory experiments, but in order to get more exact values the model output must be adjusted until it compares reasonably well with extensive field data. A review of some of the procedure for solving this type of problem, which is often called system parameter identification, form the subject of this appendix.

Only little work seems to have been reported in the literature on the systematic estimation of the empirical parameters in lake models. In a series of papers Simons (1974, 1975, 1976) considered the problem of verifying a numerical model of Lake Ontario. Since his model of the fluid flow was linear he could estimate an effective wind-stress coefficient by the comparison of observed and computed water levels and vertically mean currents. The bottom stress coefficient was taken to be equal to the wind-stress coefficient. The presented comparisons in Simons (1974) are for three different time scales: a) currents averaged over three days, b) current smoothed by a digital filter which removed all frequencies equal to or higher than the frequency of the inertial oscillation for Lake Ontario, and c) periods less than or equal to the inertial period. Only for the last time scale are the effects of varying some of the turbulent parameters presented.

In Penumalli et al (1976) estimates of the turbulent dispersion coefficients were obtained for an estuary in Texas by adjusting these coefficients in a steady state model until the least square difference between the model output and the measured field data is sufficiently small. In this study the components of the water velocity vector were prescribed and the mathematical model consisted of the linear mass balance equation.

In this report we describe some of the techniques developed in other fields of engineering for system parameters identification and unconstrained optimization and show how they can be applied to a typical model of a lake. These techniques have been developed and applied to chemical engineering, [Seinfeld (1969)], hydraulics, e.g. [Yeh (1975)] and other fields. A general review of the application of some of these techniques to control engineering is given in Astrom and Eykhoff (1971) .

The importance of the verification of a given lake model lies in the fact that if the model is not correctly verified, at best only very qualitative results can be obtained.

2. Formulation of a specific model

In order to illustrate the discussion in this appendix we will consider the depth-integrated two-dimensional model for water movements used in the Hamilton Harbour Study (1974).

Let (x,y) be a cartesian coordinate system with t being the time variable. Then the model consists of the equations of motion

$$U_t + UU_x + VU_y - fV + g\,\zeta_x + g\,\frac{U(U^2+V^2)^{\frac{1}{2}}}{C^2H} - \frac{\tau^x}{\rho H} = 0 \qquad (1)$$

$$V_t + UV_x + VV_y + fU + g\,\zeta_y + g\,\frac{V(U^2+V^2)^{\frac{1}{2}}}{C^2H} - \frac{\tau^y}{\rho H} = 0 \qquad (2)$$

and the continuity equation

$$\zeta_t + \frac{\partial}{\partial x}(HU) + \frac{\partial}{\partial y}(HV) = 0 \qquad (3)$$

where $U_x = \partial U/\partial x$, etc.

U = velocity in x direction

V = velocity in y direction

f = Coriolis parameter

g = acceleration due to gravity

C = Chezy coefficient representing bottom friction

H = depth = ζ + reference depth

ζ = elevation of water surface above reference

τ^x = component of the wind-stress in the x direction

τ^y = component of the wind-stress in the y direction

ρ = density of water.

The relationships between the wind-stress terms τ^x and τ^y and the wind are

$$\tau^x = \theta\, w^2 \sin \psi$$
$$\tau^y = \theta\, w^2 \cos \psi \tag{4}$$

where θ = wind-stress coefficient

w = wind speed

ψ = direction of the wind.

This model contains two unknown parameters: the Chezy coefficient C representing bottom friction and the wind-stress coefficient θ. Some systematic method for choosing these parameters must be used if the output of the model is to be trusted. In this paper we consider minimization of a criterion function. In the remaining part of the paper it will be assumed that a numerical procedure for solving equations (1), (2) and (3) together with appropriate boundary and initial conditions has been developed, so that accurate numerical approximations to U, V and ζ can be calculated.

3. Minimization of a criterion function

In this method the unknown parameters C and θ are chosen such that a measured variable, for example the velocity at a point, agrees

well with the calculated values of the same quantity. This raises three questions:

1) What quantities should be used for this comparison?
2) What criterion should be used to measure how well observed and calculated quantities agree?
3) How can a systematic procedure be devised for adjusting C and θ so that the criterion in 2) is satisfied?

We shall now consider these three points in turn.

1) Choice of comparison quantities.

There are basically four different types of quantities which can be used for comparison purposes (see Seinfeld (1969)).

a) Time and space dependent quantities, e.g. the elevation ζ at several points and at different times.

b) Time dependent quantities, e.g. the volume by which the lake differs from the initial volume or the velocity at a given point.

c) Space dependent quantities, e.g. the maximum elevation for the time interval of interest at each point of the lake.

d) Time and space independent quantities, e.g. the maximum elevation for the time interval of interest for all points of the lake.

If the measured data is already available, the choice of comparison quantities is made. However if the data has not been collected at the start of the program, careful thought should be given to the choice of comparison quantities within the technical limitations of the equipment available for measurement. In general one is restricted to velocity measurements at different points at regular time intervals. It is important to obtain measurement for reasonably long time period.

2) Criterion functions

The two most commonly used criterion functions are

a) the least square criterion, and

b) the minimax criterion.

a) Least square criterion

Suppose the measured data consists of the velocities U and V at a fixed point (x^*, y^*) at regular time intervals t_j, $j = 0, 1, \ldots, N$; write these measurements as

$$
\begin{aligned}
U_m(t_j) &= U(x^*, y^*, t_j) \\
V_m(t_j) &= V(x^*, y^*, t_j)
\end{aligned}
\qquad (5)
$$

The corresponding calculated quantities we denote by

$$U_C(t_j, C, \theta) \quad \text{and} \quad V_C(t_j, C, \theta);$$

these, of course, depend on the particular values of C and θ used in the calculations. We now define

$$
\begin{aligned}
D(C,\theta) = \sum_{j=1}^{N} \{ &[U_m(t_j) - U_C(t_j,C,\theta)]^2 \\
&+ [V_m(t_j) - V_C(t_j,C,\theta)]^2 \}
\end{aligned}
\qquad (6)
$$

The problem is now to find those values of C and θ for which D is a minimum, i.e.

$$\min_{C,\theta} D(C,\theta) \qquad (7)$$

This is the classical least squares criterion with uniform weights.

b) Minimax criterion

If we define

$$
\begin{aligned}
\varepsilon_j &= |U_m(t_j) - U_C(t_j,C,\theta)| \\
\eta_j &= |V_m(t_j) - V_C(t_j,C,\theta)|
\end{aligned}
\qquad (8)
$$

the minimax criterion function can be written in the form

$$\min_{C,\theta}\ \max_{j}\ (\varepsilon_j + \eta_j); \quad j = 1, 2, \ldots, N$$

here we have assumed uniform weighting functions. This can also be written in the form

$$\min_{C,\theta}\ \overline{D}(C,\theta) \tag{9}$$

where $$\overline{D}(C,\theta) = \max_{j}\ (\varepsilon_j + \eta_j)$$

$$j = 1, 2, \ldots, N$$

3) Numerical methods for calculating C and θ.

The problem of calculating C and θ is a typical unconstrained optimization problem. Many different procedures have been devised and presented in the literature. The following discussion is mainly based on Kowalik and Osborne (1968).

We shall consider three different methods and try to point out advantages and disadvantages of each method. One point which must be kept in mind is that usually it is very time-consuming to calculate $D(C,\theta)$, so it is necessary to restrict our attention to methods which converge reasonably quickly.

(i) The method of steepest descent.

Suppose we have estimates for C and θ, say $C^{(k)}$ and $\theta^{(k)}$, and we wish to improve on these estimates. We can do this by first computing the gradient of D at $C^{(k)}$ and $\theta^{(k)}$. Numerically this gradient can be approximated by

$$\nabla D\Big|_{\substack{C^{(k)}\\ \theta^{(k)}}} = \hat{i}\ \frac{D(C^{(k)},\theta^{(k)}) - D(C^{(k-1)},\theta^{(k)})}{C^{(k)} - C^{(k-1)}} + \hat{j}\ \frac{D(C^{(k)},\theta^{(k)}) - D(C^{(k)},\theta^{(k-1)})}{\theta^{(k)} - \theta^{(k-1)}}$$

where $\hat{i}$ and $\hat{j}$ are mutually perpendicular unit vectors.

Thus we can write the new values of C and θ as

$$C^{(k+1)} = C^{(k)} + \alpha\hat{i} \cdot \nabla D$$

$$\theta^{(k+1)} = \theta^{(k)} + \alpha\hat{j} \cdot \nabla D$$

where α is a positive parameter. The question is now how to choose α. One way is to pick α so that

$$D(C^{(k+1)}, \theta^{(k+1)})$$

is a minimum. It is also possible to give α a constant value at the beginning of the procedure and then use this value for each iteration.

The advantage of the method of steepest descent is its simplicity; it is easy to understand and to apply. However, its performance is many times disappointing; after a few satisfactory iterations the convergence rate become very small. It is not much used anymore.

(ii) The method of Davidon, Fletcher and Powell.

This method, which is also called a variable metric method, seems to be the best general-purpose optimization procedure making use of derivatives that is currently available. It is a modified gradient method; it uses certain information which is generated at each iteration to construct the Hessian matrix of the function. This indicates that the method attempts to use second-order information while the method of steepest descent uses only first-order information.

In terms of a general function $F(\underline{x})$ where $\underline{x}$ is the vector $(x_1, x_2, \ldots, x_n)$ the i-th stage of the algorithm can be expressed as follows.

Compute $\quad \underline{d}_i = -H_{i-1}\, \underline{g}_i$

compute $\quad \lambda_i$ to minimze $F(\underline{x}^{(i)} + \lambda \underline{d}_i)$

set $\quad \underline{x}^{(i+1)} = \underline{x}^{(i)} + \lambda_i\, \underline{d}_i$

compute $\quad \underline{g}_{i+1} = \nabla F(\underline{x}^{(i+1)})$

set $\quad \underline{y}_i = \underline{g}_{i+1} - \underline{g}_i$

compute $\quad H_i = H_{i-1} + \lambda_i \dfrac{\underline{d}_i \, \underline{d}_i^{\,T}}{\underline{g}_i^T \, H_{i-1} \, \underline{g}_i}$

Here $\underline{d}_i^T$ is the transpose of $\underline{d}$. The first stage of the algorithm consists of choosing a starting point $\underline{x}^{(0)}$ and an initial approximation H_0 for the Hessian matrix; it is customary to set

$$H_0 = I$$

where I is the unit matrix. It is also necessary to calculate

$$\underline{g}_0 = \nabla F(\underline{x}^{(0)}).$$

For the particular problem we are concerned with $n = 2$ and $x_1 = C$, $x_2 = \theta$ and

$$F(\underline{x}) = D(C, \theta).$$

The advantage of this method is its fast convergence rate, considerably faster than the method of steepest descent. However, it does require the calculation of the gradient of D or an approximation to it which can be time-consuming.

(iii) Powell's method without derivatives.

In contrast to the two methods discussed above this method does not require the calculation of the gradient of D. It is based on the idea of conjugate directions. Again we will present the method for a general function $F(\underline{x})$ where $\underline{x}$ is the vector $(x_1, x_2, \ldots, x_n)$. We assume that we initially have a starting point $\underline{x}^{(0)}$ and n independent vectors $\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n$. Then the ith stage is

Compute $\quad \lambda_0$ to minimze $\; F(\underline{x}^{(0)} + \lambda \underline{d}_n)$

set $\underline{x}^{(1)} = \underline{x}^{(0)} + \lambda_0 \underline{d}_n$

For $j = 1, 2, \ldots, n$ compute λ_j to minimize

$$F(\underline{x}^{(j)} + \lambda \underline{d}_j)$$

and set $\underline{x}^{(j+1)} = \underline{x}^{(j)} + \lambda_j \underline{d}_j$

set $\underline{d}_j = \underline{d}_{j+1}$, $j = 1, 2, \ldots, n-1$

set $\underline{d}_n = \underline{x}^{(n+1)} - x^{(n)}$

set $\underline{x}^{(0)} = \underline{x}^{(n+1)}$

Go back to the beginning.

This method usually requires more iterative steps before convergence is attained than the method of Davidon, Fletcher and Powell. However since it is not necessary to evaluate the gradient of D at each step, the total computing time required is often similar.

For all three methods it is necessary to have a criterion for terminating the calculations. Usually it is that

$$D(C^{(n+1)},\theta^{(n+1)}) - D(C^{(n)},\theta^{(n)}) < \varepsilon$$

where ε is some preassigned tolerance. A reasonable value is often

$$\varepsilon = 10^{-3}.$$

4. Conclusions

The desired sequence of steps in producing a reliable numerical model can now be stated as follows:

a) sufficient field data is collected so that the main physical processes in the lake can be stated.

b) an adequate mathematical model containing various empirical parameters is formulated.

c) a numerical procedure for solving this model is devised.

d) field data is collected so that adequate boundary and initial conditions for the model are obtained.

e) numerical experiments are carried out to ensure that the time and space step sizes are sufficiently small.

f) adequate field data which is compatible with the model, is collected.

g) a procedure, for example one of the procedures described in this report, is used to calculate reliable values of the different empirical parameters in the model.

It is now possible to use the model with a fair amount of confidence to predict quantitatively the effects of several different technical solutions to water quality management problems. However, without the validation of the model being carried out usually only qualitative results can be expected.

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Appendix B

Application to the Hamilton Harbour model of some methods for the validation of lake models.

1. Introduction

Some of the procedures for model validation outlined in Appendix A were applied to the results obtained using the numerical model of Hamilton Harbour developed by the Water Resources Branch, Ontario Ministry of the Environment; see the Hamilton Harbour Study (1974). The results and the conclusions that can be drawn from them are presented in this appendix. The measured data used in these calculations were supplied by the Water Resources Branch. The measured and calculated data were obtained for grid point (15,6).

2. Formulation

The model being considered in this appendix is essentially the same as the model described in Appendix A except that the Chezy coefficient C which models bottom friction is replaced by

$$C = \frac{1.49}{n} H^{\frac{1}{6}} .$$

The parameter n, called Manning's n, can be assigned different values in different parts of the lake; in this study four different values of n, denoted by n_1, n_2, n_3 and n_4, were used. The numerical solution of the model was obtained using a program developed by the Water Resources Branch, Ontario Ministry of the Environment.

3. Results

Several solutions were obtained corresponding to different values of the wind stress parameter θ and different sets of values for n_1, n_2, n_3 and n_4, Manning's n. Since the n's were changed by the same percentage only the value of n_1 is given for each solution; the base values are $n_1 = .016$, $n_2 = .023$, $n_3 = .030$ and $n_4 = .040$.

In Table 1 we present some results for 12 hours of real time using a time step of 20 sec. It is seen that the lowest value of the least square criterion is obtained at Runs 5 and 8 and the highest value at Run 15. Some of these results are also presented in Figure 1. This seems to indicate that the minimum of the least square criterion is in the neighbourhood of θ = .0032 and n_1 = .0126 and that this minimum is not much less than 0.2374. Without extensive and very expensive computation it is not possible to state conclusively that this minimum is the smallest that can be obtained; i.e. it would be very expensive to prove that this minimum is global and not local.

Some results for the max-min criterion are also given in Table 1 and we see that the minimum is in the neighbourhood of θ = .0035 and n_1 = .0126 with a minimum less than .0842. Again it would be very expensive to prove that this minimum is global and not local.

Finally in Table 2 we give the mean values of the two velocity components. We see that only in Run 1 is one of the velocity components in the wrong direction.

From Table 1 we see that the optimum values for θ and n_1 are the ones used in Runs 5, 8 and 11. In order to analyse these runs in more detail we calculated the percentage differences between the mean and the root mean square of the measured velocities and the model output; these are presented in Table 3.

4. Conclusions

Several conclusions can be drawn from the results presented in the previous section.

a) Major features seem to be modelled reasonably well. This is indicated by the fact that
   (i) For all runs except the first one the mean velocities are in the right directions.

(ii) The differences in the rms values of the measured and calculated data are not too large.

b) With the present time step size and space mesh, the detailed features are not modelled as accurately as the major features; this conclusion is based on

(i) The least square and min-max criteria are only satisfied very approximately.

(ii) The difference in the rms values of the measured and calculated V velocities is around 50%.

c) The best values for θ and n_1 seem to lie in the neighbourhood of .0035 and .0126, respectively. However, Table 1 indicates that the two criteria give slightly different optimum values.

It has been supposed the windstress parameter θ does not vary from grid point to grid point but is constant. It is known that this is usually not the case, but since very little is known about this spatial variation of θ, it is difficult to incorporate this into the mathematical model. However the fact that it is not done may be one of the reasons why the detailed features of the flow in the harbour are not as well modelled as the major features.

References

Hamilton Harbour Study (1974). Report prepared by the Water Quality Branch, Ontario Ministry of the Environment.

Table 1: Least square and max criteria for different values of θ and n_1.

Run #	Windstress coefficient	Manning's n_1	$\sum[(U_m - U_c)^2 + (V_m - V_c)^2]$	$\mathrm{Max}[\lvert U_m - U_c\rvert + \lvert V_m - V_c\rvert]$
1	.0020	.012544	.3614	.1034
2	.0025	.012567	.2951	.0991
3	.0030	.016000	.2575	.0952
4	.0030	.012609	.2447	.0932
5	.0032	.012600	.2374	.0898
6	.0032	.011970	.2410	.0892
7	.0032	.012240	.2388	.0894
8	.0032	.013000	.2374	.0901
9	.0032	.013500	.2386	.0906
10	.0034	.012591	.2493	.0861
11	.0035	.012600	.2638	.0842
12	.0040	.013400	.3387	.1043
13	.0040	.011970	.4217	.1282
14	.0040	.012559	.3844	.1205
15	.0050	.012508	.5360	.1349

Table 2: Mean values of u and v for a 12 hour period.

Run #	Mean value of U	Mean value of V
1	.01150	.00636
2	.01601	-.00495
3	.02191	-.01147
4	.02193	-.01430
5	.02329	-.01574
6	.02277	-.01463
7	.02302	-.01517
8	.02353	-.01617
9	.02378	-.01643
10	.02380	-.01554
11	.02376	-.01499
12	.02421	-.01331
13	.01855	-.00864
14	.02133	-.01043
15	.01620	-.00366
Measured data	.00712	-.02199

Table 3: Comparison of Runs 5, 8 and 11.

Run	5	8	11
θ	.0032	.0032	.0035
n_1	.0126	.0130	.0126
Least square criterion	.2374	.2374	.2638
Min-max criterion	.0898	.0901	.0842
Difference in mean for U	227%	230%	234%
Difference in rms for U	54%	54%	35%
Difference in mean for V	28%	26%	32%
Difference in rms for V	9%	10%	22%

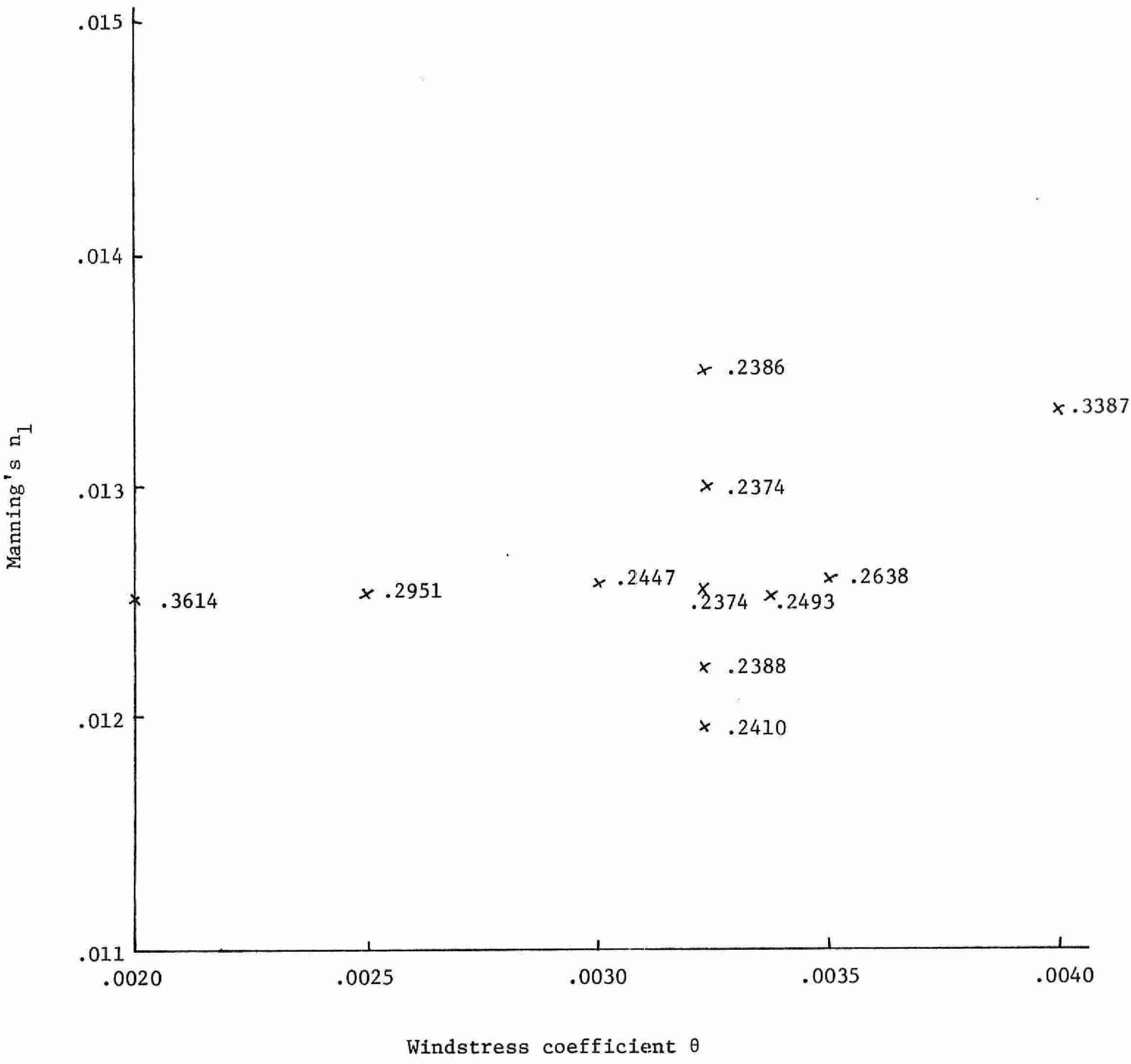

Figure 1. Values of $\sum[(U_m - U_c)^2 + (V_m - V_c)^2]$.

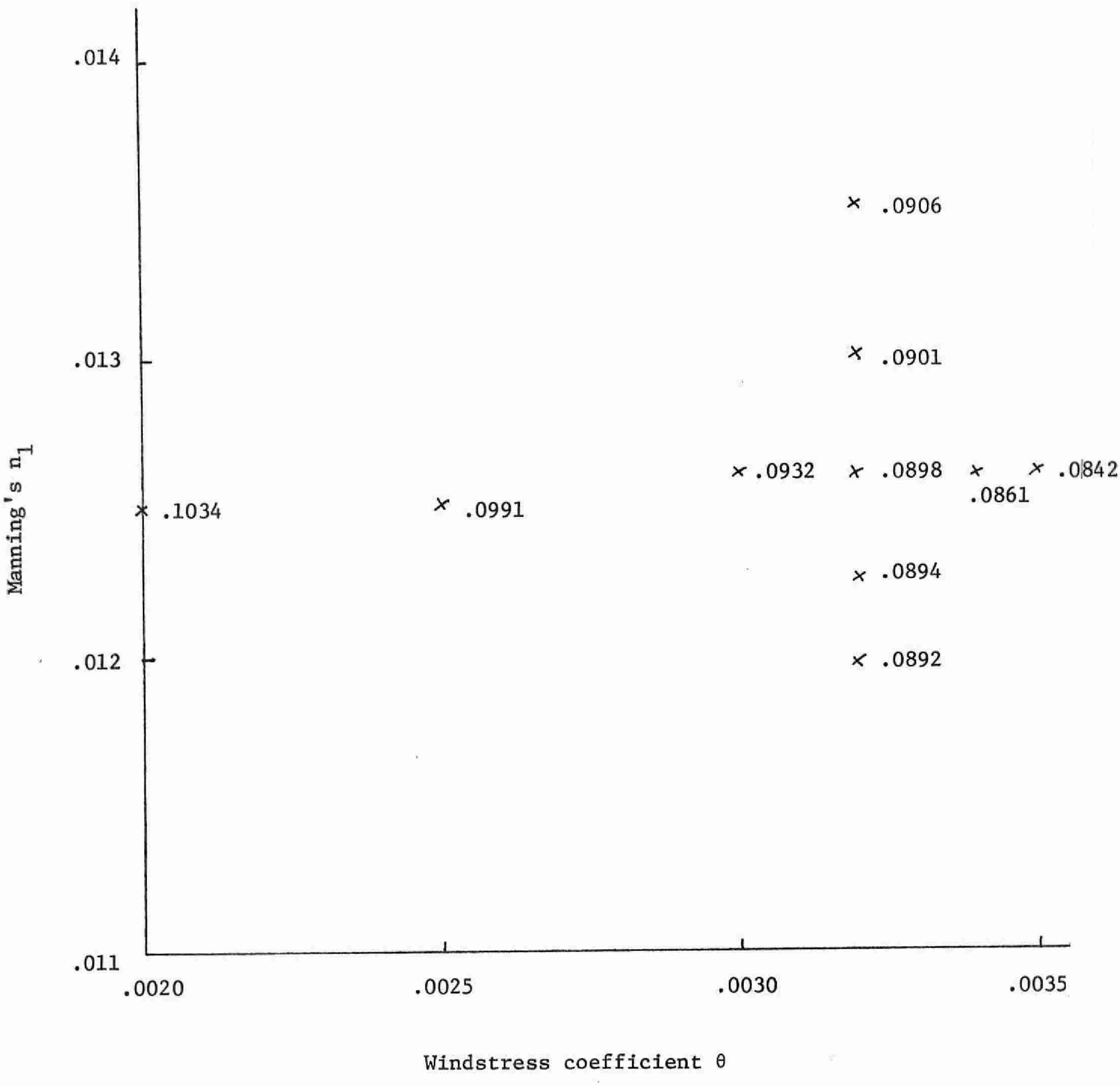

Figure 2. Values of $\text{Max}[|U_m - U_c| + |V_m - V_c|]$

Appendix C

On the boundary conditions for wind-driven lake circulation models.*

1. Introduction

Approximate solutions for the steady wind-driven circulation in shallow lakes have been calculated by Hadjitheodorou (1) and Liggett and Hadjitheodorou (2) using an Ekman-type model based on the assumption of small Rossby number and small horizontal turbulent mixing. Later this analysis was extended by Gedney and Lick (3). A review of this type of model is given in Cheng, Powell and Dillon (4).

The model can be expressed as a second order linear elliptic partial differential equation for a volumetric-transport stream-function ψ. In (2), (3) and (4) the boundary condition imposed on ψ is that ψ is constant on all boundaries. In this paper we show that in addition to this condition, the boundary conditions on the velocity components imply that the normal derivative of ψ must also vanish on the boundaries. However, for a second order elliptic equation both of these conditions cannot be satisfied simultaneously. It is shown in the last section of the paper how the model can be modified so as to remove this difficulty. From this we can draw the conclusion that this type of model is not adequate near the shore of a lake. In this region a more complete model containing all the viscous terms must be used.

2. Formulation

Let (x,y,z) and (u,v,w) be the nondimensional position and velocity vectors, respectively, in a rectangular coordinate system with x positive eastward, y positive northward, and z positive upward and zero at the surface. If p is the nondimensional pressure, the

* This appendix has been written together with Mr. P. Forsyth, Jr., Department of Applied Mathematics, University of Western Ontario.

linearized equations for the wind-driven lake circulation are

$$-v = -p_x + \frac{1}{2m^2} u_{zz} \tag{1}$$

$$u = -p_y + \frac{1}{2m^2} v_{zz} \tag{2}$$

$$p_z = -1 \tag{3}$$

$$u_x + v_y + w_z = 0 \tag{4}$$

where $m^2 = \frac{fD^2}{2\eta}$

f = Coriollis parameter (assumed to be constant)

D = typical vertical dimension

η = eddy viscosity

p_x = $\partial p/\partial x$, etc.

These equations are derived in Liggett and Hadjitheodorou (2).

The boundary conditions are

$$u_z = \frac{fL\tau_x}{\eta g} = \Delta, \quad v_z = \frac{fL\tau_y}{\eta g} = \Gamma, \quad w = 0 \text{ at } z = 0 \tag{5}$$

$$u = v = w = 0 \quad \text{at} \quad z = -h(x,y) \tag{6}$$

where

z = $-h(x,y)$ is the bottom of the lake

L = typical horizontal dimension

τ_x = wind stress in the x direction

τ_y = wind stress in the y direction

g = gravitational attraction

3. Analysis

Equation (3) implies that p_x and p_y are independent of z, so equations (1) and (2) can be integrated to give

$$u = -p_y + \cos mz(C_2e^{mz} - C_4e^{-mz})$$

$$- \sin mz(C_1e^{mz} - C_3e^{-mz}) \tag{7}$$

$$v = p_x + \cos mz(C_1e^{mz} + C_3e^{-mz})$$

$$+ \sin mz(C_2e^{mz} + C_4e^{-mz}) \tag{8}$$

where C_1, C_2, C_3 and C_4 are functions of x and y which are evaluated by imposing the boundary conditions (5) and (6); they are given in the appendix. The pressure p is still unknown.

An additional equation is obtained by the integration of the continuity equation (4) with respect to z from z = -h to z = 0; hence

$$\int_{-h}^{0} u_x \, dz + \int_{-h}^{0} v_y \, dz = 0 \tag{9}$$

which can also be written in the form

$$\frac{\partial}{\partial x} \int_{-h}^{0} u dz + \frac{\partial}{\partial y} \int_{-h}^{0} v dz = 0 \tag{10}$$

since u = v = w = 0 at z = -h and w = 0 at z = 0. This equation is satisfied by the function $\psi(x,y)$ defined by

$$\psi_x = -\int_{-h}^{0} v dz \text{ and } \psi_y = \int_{-h}^{0} u dz. \tag{11}$$

If we substitute u and v from equations (7) and (8) into the above expressions and integrate, we get

$$\psi_x = hh_4p_x - hh_1p_y - hh_3\Gamma + hh_2\Delta \tag{12}$$

$$\psi_y = hh_1p_x + hh_4p_y + hh_2\Gamma + hh_3\Delta \tag{13}$$

where h_1, h_2, h_3 and h_4 are functions of x and y given in the appendix. These expressions can be solved for p_x and p_y, and since

$$p_{xy} = p_{yx} \tag{14}$$

the following equation for ψ is obtained

$$\nabla^2\psi = \frac{h}{h_1}(h_1^2 + h_4^2)[(r_x + s_y)\psi_x + (r_y + s_x)\psi_y + \frac{\partial}{\partial x}(q\Gamma + t\Delta) + \frac{\partial}{\partial y}(t\Gamma - q\Delta)] \tag{15}$$

where q, r, s and t are functions of h, h_1, h_2, h_3 and h_4 given in the appendix.

The analysis so far is similar to the one presented in Liggett and Hadjitheodorou (2). However the treatment of the boundary conditions on ψ is different. In order to simplify the subsequent analysis we shall consider a rectangular lake with boundaries parallel to either the x or y axes. Consider now one of the boundaries which we can assume to be along y = 0. As y approaches zero, we see from equation (11) that ψ_x and ψ_y must also approach zero, since u and v must be finite, i.e.

$$\psi_x = \psi_y = 0 \text{ at } y = 0 \tag{16}$$

These conditions, which also hold at the other boundaries, are equivalent to

$$\psi = C, \ \psi_n = 0 \text{ at } y = 0 \tag{17}$$

where ψ_n denotes the normal derivative of ψ and C is a constant. If there is no inflow into the lake, we can take C to be zero. Liggett and Hadjitheodorou (2) and Gedney and Lick (3) only used the first of these conditions.

Since equation (15) for ψ is a second order elliptic partial differential equation it is clear that we can only impose one of the two boundary conditions in (17). It is easy to see how this problem of too many boundary conditions arose. In the derivation of the linearized equations (1) and (2) it was assumed that the horizontal viscous terms were much smaller than the vertical viscous terms and hence could be ignored. This effectively lowered the order of the equation; however, the complete set of boundary conditions (5) and (6) were retained.

4. Treatment of the boundary conditions.

In Liggett and Hadjitheodorou (2) and Gedney and Lick (3) it was implicitly assumed that the condition $\psi_n = 0$ on the boundaries could be ignored. However, as we can see from the definition of ψ, equations (11), this may lead to u and v being infinite at the boundaries. A more suitable treatment of the boundary conditions consists of supposing that the model defined by equations (15) and (17) is only valid in the interior of the lake. Along the shore there is a region in which a more complicated model is required.

Let us now illustrate how this idea can be used in a numerical procedure. Suppose we have a rectangular lake with sides parallel to the x and y axes; say $x = 0$, $x = X$, $y = 0$ and $y = Y$ where X and Y are given constants. We cover the lake by a mesh defined by

$$x_i = i\,\Delta x,\; y_j = j\,\Delta y \tag{18}$$

where $i = 0, 1, 2, \ldots, X/\Delta x$, and $j = 0, 1, 2, \ldots, Y/\Delta y$. Consider now the region near $y = 0$, see Figure 1. At $y = 0$ $\psi = C$ and $\psi_y = 0$. The last condition can be approximated by

$$\frac{\psi_{i,1} - \psi_{i,0}}{\Delta y} = 0 \tag{19}$$

to order Δy. Hence

$$\psi_{i,1} = \psi_{i,0} = C; \tag{20}$$

similar conditions can be obtained near the other boundaries. We have now a properly posed problem for the region

$$1 < i < \frac{X}{\Delta x} - 1, 1 < j < \frac{Y}{\Delta y} - 1 \tag{21}$$

The difference between this problem and the problem solved by Liggett and Hadjitheodorou (2) and Gedney and Lick (3) is that here the depth function $-h$ is not zero at the point where ψ is equal to a constant. Since the grid spacings used in most finite

difference treatments of lake models are fairly large, e.g. Gedney and Lick (3) used $x = y = 0.5$ miles in some parts and $x = y = 2$ miles in other parts, this difference can lead to large differences in the calculated current patterns, especially near the shore.

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Appendix

It is convenient to define several intermediate expressions due to the length of the equations. First, the following are defined:

$$\alpha = \cos^2 mh(e^{-mh} + e^{mh})^2 + \sin^2 mh(e^{-mh} - e^{mh})^2 \tag{22}$$

$$\beta = \frac{1}{2m} e^{-mh} (\sin mh - \cos mh) \tag{23}$$

$$\gamma = \sin mh(e^{-mh} - e^{mh}) \tag{24}$$

$$\delta = \frac{1}{2m} e^{-mh}(\sin mh + \cos mh) \tag{25}$$

$$\varepsilon = \cos mh(e^{-mh} + e^{mh}) \tag{26}$$

$$\kappa = \frac{1}{2m} e^{mh}(\sin mh + \cos mh) \tag{27}$$

$$\lambda = \frac{1}{2m} e^{mh}(\sin mh - \cos mh) \tag{28}$$

Then the expressions for C_1, C_2, C_3 and C_4 can be written in the form

$$C_1 = C_3 + \frac{\Gamma}{2m} - \frac{\Delta}{2m}$$

$$C_2 = - C_4 + \frac{\Gamma}{2m} + \frac{\Delta}{2m}$$

$$C_3 = \frac{1}{\alpha} [- \varepsilon \frac{\partial p}{\partial x} + \gamma \frac{\partial p}{\partial y} + \Gamma(\beta\varepsilon - \delta\gamma) + \Delta(\beta\gamma + \delta\varepsilon)] \tag{29}$$

$$C_4 = \frac{1}{\alpha} [- \gamma \frac{\partial p}{\partial x} - \varepsilon \frac{\partial p}{\partial y} + \Gamma(\beta\gamma + \delta\varepsilon) - \Delta(\beta\varepsilon + \delta\gamma)] \tag{30}$$

The coefficients in equations (12) and (13) are

$$h_1 = \frac{1}{\alpha h} [\gamma(\beta + \kappa) + \varepsilon(\delta + \lambda)] \tag{31}$$

$$h_2 = \frac{1}{\alpha h} [\frac{\alpha(\frac{1}{m} + \beta - \delta)}{2m} - (\beta\gamma + \delta\varepsilon)(\beta + \kappa) - (\beta\varepsilon - \delta\gamma)(\delta + \lambda)] \tag{32}$$

$$h_3 = \frac{1}{\alpha h} [\frac{\alpha(\beta + \delta)}{2m} + (\beta\varepsilon - \delta\gamma)(\beta + \kappa) - (\beta\gamma + \delta\varepsilon)(\delta + \lambda)] \tag{33}$$

$$h_4 = \frac{1}{\alpha h} [- \alpha h + \varepsilon(\beta + \kappa) - \gamma(\delta + \lambda)] \tag{34}$$

The coefficients in equation (15) are

$$r = \frac{h_1}{h(h_1^2 + h_4^2)}, \quad s = \frac{h_4}{h(h_1^2 + h_4^2)}, \quad q = \frac{h_1 h_3 + h_2 h_4}{h_1^2 + h_4^2}, \quad t = \frac{h_3 h_4 - h_1 h_2}{h_1^2 + h_4^2} \tag{35}$$

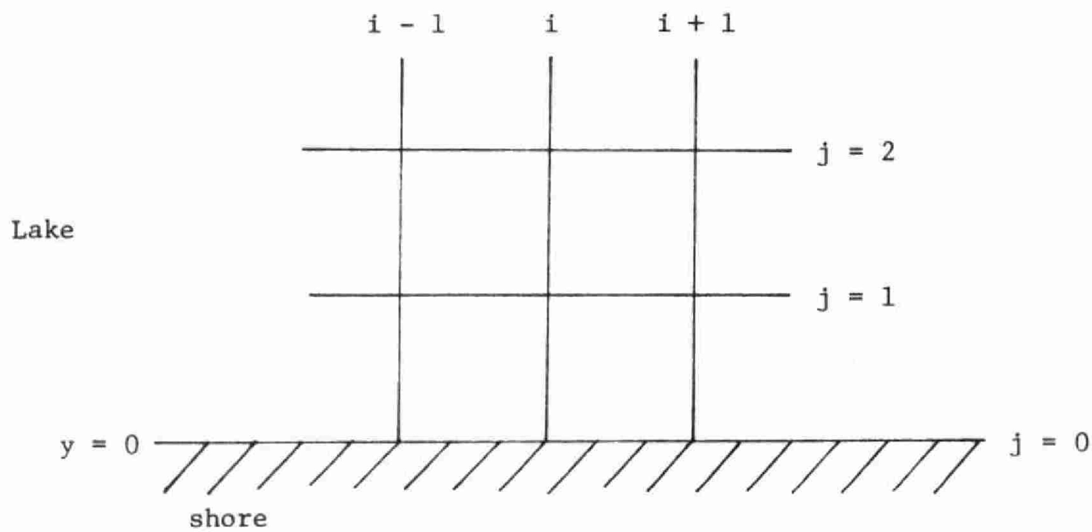

Figure 1. Finite difference mesh near the shore.